1. This first problem is intended to give you practice evaluating the divergence and curl of simple vector fields. Hopefully, it will simultaneously help develop some intuition which will allow you to anticipate when to expect the result to zero/nonzero.

(a) Assume that $\mathbf{v} = x\,\hat{x}$.

(i) Plot the vector field.
(ii) Do you expect $\nabla \cdot \mathbf{v}$ to be zero or nonzero? Why?
(iii) Evaluate $\nabla \cdot \mathbf{v}$.
(iv) Do you expect $\nabla \times \mathbf{v}$ to be zero or nonzero? Why?
If it's nonzero, what is the direction of $\nabla \times \mathbf{v}$?
(v) Evaluate $\nabla \times \mathbf{v}$.

(b) Assume that $\mathbf{v} = y\,\hat{x}$.

(i) Plot the vector field.
(ii) Do you expect $\nabla \cdot \mathbf{v}$ to be zero or nonzero? Why?
(iii) Evaluate $\nabla \cdot \mathbf{v}$.
(iv) Do you expect $\nabla \times \mathbf{v}$ to be zero or nonzero? Why?
If it's nonzero, what is the direction of $\nabla \times \mathbf{v}$?
(v) Evaluate $\nabla \times \mathbf{v}$. Don't forget to specify the direction.

2. Consider the scaler function $T = \dfrac{3x^2y}{z}$.

(a) Find the gradient of $T$: $\nabla T$.
(b) Construct a unit vector in the direction of the steepest descent at the point $(x, y, z) = (1, 1, 1)$.
(c) Find any unit vector that is perpendicular to $\nabla T$ at the point $(x, y, z) = (1, 1, 1)$.

3. Prove the vector triple product identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$$

by explicitly evaluating the cross products in Cartesian coordinates.

4. In class, we proved that the divergence of the electric field due to a point charge $q$ is zero everywhere expect at the exact position of $q$. We did this by expressing the electric field in spherical coordinates $\left(\mathbf{E} = \frac{k_e q}{r^2}\,\hat{r}\right)$ and then evaluating the divergence $\nabla \cdot \mathbf{E}$ in spherical coordinates. In this problem, I want you to repeat this problem in Cartesian coordinates. Of course, you should still get the same answer!

(a) This problem is easiest to do if one first realizes that, in spherical coordinates, we can express the electric field of a point charge as:

$$\mathbf{E} = k_e q \frac{\mathbf{r}}{r^3}$$

which then allows us to easily transform into Cartesian coordinates. Write down the electric field in terms of Cartesian coordinates.

(b) Now evaluate the divergence in Cartesian coordinates.

5. In class, we proved that the curl of the magnetic field due to a long, straight current is zero everywhere expect at the exact position of $I$. We did this by expressing the magnetic field in cylindrical coordinates $\left(\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}\right)$ and then evaluating the curl $\nabla \times \mathbf{B}$ in cylindrical coordinates. In this problem, I want you to repeat this problem in Cartesian coordinates. Of course, you should still get the same answer!

(a) Start by showing that in cylindrical coordinates:

$$\begin{aligned}\hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ &= -\frac{y\,\hat{x}}{\sqrt{x^2+y^2}} + \frac{x\,\hat{y}}{\sqrt{x^2+y^2}}.\end{aligned}$$

(b) Next, show that, in Cartesian coordinates, the magnetic field due to a long, straight current is expressed as:

$$B = \mu_0 I \left(\frac{-y\,\hat{x} + x\,\hat{y}}{x^2+y^2}\right).$$

(c) Finally, evaluate $\nabla \times \mathbf{B}$ in Cartesian coordinates.